v15 #2

Scanned

from sixty to two

THE EVOLUTION OF NUMBER SYSTEMS

by DONALD E. KNUTH

This is a portion of Volume 2 of a seven-volume series titled *The Art of Computer Programming*, to be published by Addison-Wesley. The author's intent is to explain and illustrate in the se[illegible]ost of what is known about basic computer program[illegible]chniques exclusive of numerical analysis. Several thousa[illegible]xercises (with answers) will be included. The project was begun in 1962. Volume 3 is expected to be published next year.

The historical development of number representations is a fascinating story, since it parallels the development of civilization itself. The advent of computers has focussed considerable attention on num[illegible] systems whose radix (or base) is different from ten. [illegible] purpose of this article is to examine the origins of the[illegible]tems, together with a discussion of significant milest[illegible] in their development. The earliest forms of number representations, still found in primitive cultures, are generally based on groups of fingers, or piles of stones, etc., usually with special conventions about replacing a larger pile or group of, say, five or ten objects by one object of a special kind or in a special place. Such systems lead naturally to the earliest ways of representing numbers in written form, such as the systems of Babylonian, Egyptian, Greek, Chinese, and Roman numerals, but these notations are quite [illegible]venient for performing arithmetic operations except i[illegible] simplest cases.

Dur[illegible] the twentieth century, historians of mathematics have made extensive studies of early cuneiform tablets found by archeologists in the Middle East. These studies show that the Babylonian people actually had two distinct systems of number representation: Numbers used in everyday business transactions were written in a notation based on grouping by tens, hundreds, etc.; this notation was inherited from earlier Mesopotamian civilizations, and large numbers were seldom required. When more difficult mathematical problems were considered, however, Babylonian mathe[illegible]cians made extensive use of a sexagesimal (radix 60) p[illegible]onal notation which was highly developed at least as early as 1750 B.C. This notation was unique in that it was actually a *floating-point* form of representation with exponents omitted: the proper scale factor or power of 60 was to be supplied by the context, so that, for example, the numbers 2, 120, 7200, $\frac{1}{30}$, etc., were all written in an identical manner. This notation was especially convenient for multiplication and division, using auxiliary tables, since radix-point alignment had no effect on the answer; the same idea is applied today in the use of slide rules. As examples of this Babylonian notation, consider the following excerpts from early tables: The square of 30 is 15 (which may also be read, "the square of $\frac{1}{2}$ is $\frac{1}{4}$"); the reciprocal of $81 = (1\ 21)_{60}$ is $(44\ 26\ 40)_{60}$; and the square of the latter is $(32\ 55\ 18\ 31\ 6\ 40)_{60}$. The Babylonians had a symbol for zero, but because of their "floating-point" philosophy, it was used only within numbers, not at the right end to denote a scale factor.

decimal notation

Fixed-point positional notation was apparently developed first by the Maya Indians in Central America about 2000 years ago, but their number system had no important influence on the rest of the world. Furthermore, they used a mixed radix system, alternating between radix 20 and radix 18, and so it was not very suitable for operations like multiplication of large numbers, nor is there any known evidence that Mayans were skilled at arithmetic.

Several centuries before Christ, the Greek people employed an early form of the abacus to do their arithmetical calculations, using sand and/or pebbles on a board which had rows or columns corresponding in a natural way to our

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decimal system. It is perhaps surprising to us that the same positional notation was never adapted to written forms of numbers, since we are so accustomed to reckoning with the decimal system using pencil and paper; but the greater ease of calculating by abacus (since handwriting was not a common skill, and since abacus calculation makes it unnecessary to memorize addition and multiplication tables) probably made the Greeks feel it would be silly even to suggest that reckoning could be done better on "scratch paper." At the same time Greek astronomers did make use of a sexagesimal positional notation for fractions, which they learned from the Babylonians.

Our decimal notation, which differs from the more ancient forms primarily because of its fixed radix point, together with its symbol for zero to mark an empty position, was developed first in India among the Hindu people. The exact date when this notation first appeared is quite uncertain; about 600 A.D. seems to be a good guess. Hindu science was rather highly developed at that time, particularly in astronomy. The earliest known Hindu manuscripts which show this notation have numbers written backwards (with the most significant digit at the right), but soon it became standard to put the most significant digit at the left.

About 750 A.D., the Hindu principles of decimal arithmetic were brought to Persia, as several important works were translated into Arabic. Not long after this, al-Khowârizmî wrote his Arabic textbook on the subject. This text was so popular, our word "algorithm" has been derived from al-Khowârizmî's name. His book was translated into Latin and was a strong influence on Leonardo Pisano (Fibonacci), whose book on arithmetic (1202 A.D.) played a major role in the spreading of Hindu-Arabic numerals into Europe. It is interesting to note that the left-to-right order of writing numbers was unchanged during these two transitions from Hindu to Arabic and from Arabic to Latin, although Arabic is written from right to left while Hindu and Latin are written from left to right. A detailed account of the subsequent propagation of decimal numeration and arithmetic into all parts of Europe during the period from 1200 to 1600 A.D. is given by David Eugene Smith in his *History of Mathematics* I (Boston: Ginn and Co., 1923), Chapters 6 and 8.

Decimal notation was applied at first only to integer numbers, not to fractions. Arabic astronomers, who required fractions in their star charts and other tables, continued to use the notation of Ptolemy (the famous Greek astronomer) which was based on sexagesimal fractions. This system still survives today, in our trigonometric units of "degrees, minutes, and seconds," and also in our units of time, as a remnant of the original Babylonian sexagesimal notation. Early European mathematicians also used sexagesimal fractions when dealing with noninteger numbers; for example, Fibonacci gave the value

$$1^\circ\ 22'\ 7''\ 42'''\ 33^{\text{iv}}\ 4^{\text{v}}\ 40^{\text{vi}}$$

as an approximation to the root of the equation $x^3 + 2x^2 + 10x = 20$.

The use of decimal notation also for tenths, hundredths, etc., in a similar way seems to be a comparatively minor change; but, of course, it is hard to break with tradition, and sexagesimal fractions have an advantage over decimal fractions in that numbers such as ⅓ can be expressed exactly in a simple way. The first known occurrence of decimal fractions dates from the 15th century, over 600 years after decimal notation for integers had been in use by the Arabs. It appeared without fanfare in a short treatise on arithmetic and geometry by Jemshîd ibn Mes'ûd al-Kashî, who died c. 1436. His remarkable work (written in Persian) gives the value of π as

integer
3 1415926535898732

which is correct to 16 decimal places. Neither the concept of decimal fractions nor such an accurate approximation to π were known in Europe until over a century later. A little known arithmetic text by Francesco Pellos (1492) made use of a "decimal point" in a completely modern manner but only for intermediate results during a calculation when dividing by a power of ten; the final answer was rewritten as a fraction. This idea had previously appeared in the writings of Regiomontanus, about 30 years earlier, who used a vertical bar instead of a decimal point. In 1525, Christoff Rudolff of Germany discovered decimal fractions for himself, but his work did not become well known. Simon Stevin of Belgium independently thought of decimal fractions in 1585, and he wrote an arithmetic text which explicitly set forth the associated theory for the first time. His work, and Napier's discovery of logarithms soon afterwards, made decimal fractions very common in Europe during the 17th century.

binary system

The binary system of notation has its own interesting history. Many primitive tribes in existence today are known to use a binary or "pair" system of counting (making groups of two instead of five or ten), but they do not count in a true radix 2 system, since they do not treat powers of 2 in a special manner. Another "primitive" example of an essentially binary system is the conventional musical notation for expressing rhythms and durations of time.

The Rhind papyrus, which is one of the first nontrivial mathematical documents known (Egypt, c. 1650 B.C.), uses a decimally oriented scheme of notation for numbers, but it shows how to perform multiplication operations by successive doubling and adding. This device is inherently based on the binary representation of the multiplier, although the binary system was not specifically pointed out.

Nondecimal number systems were discussed in Europe during the seventeenth century. For many years astronomers had occasionally used sexagesimal arithmetic both for the integer and the fractional parts of numbers, primarily when performing multiplication. The fact that *any* positive number could serve as radix was apparently first stated in print by Blaise Pascal in *De numeris multiplicibus*, which was written about 1658. Pascal wrote, "Denaria enim institute hominum, non ex necessitate naturae ut vulgus arbitratur, et sane satis inepte, posita est"; i.e., "The decimal system has been established, somewhat foolishly to be sure, according to man's custom, not from the natural necessity as most people would think." He stated that the duodecimal (radix 12) system would be a welcome change, and he gave a rule for testing a duodecimal number for divisibility by 9. Erhard Wiegel proposed the quaternary (radix 4) system in a number of publications beginning in 1673. A detailed discussion of radix 12 arithmetic was given by Joshua Jordaine, in his book *Duodecimal Arithmetick* (London, 1687).

Although decimal notation was almost exclusively used for arithmetic during that era, other systems of weights and measures were rarely if ever based on multiples of 10, and many business transactions required a good deal of skill in adding quantities such as pounds, shillings, and pence. For centuries, merchants had therefore learned to compute sums and differences of quantities expressed in peculiar units of currency, weights, and measures; and this was actually arithmetic in a nondecimal number system. The common units of liquid measure in England, dating from the 13th century or earlier, are particularly noteworthy:

2 gills	=1	chopin
2 chopins	=1	pint
2 pints	=1	quart
2 quarts	=1	pottle
2 pottles	=1	gallon
2 gallons	=1	peck
2 pecks	=1	demibushel
2 demibushels	=1	bushel or firkin
2 firkins	=1	kilderkin
2 kilderkins	=1	barrel
2 barrels	=1	hogshead
2 hogsheads	=1	pipe
2 pipes	=1	tun

Quantities of liquid expressed in gallons, pottles, quarts, pints, etc., were essentially written in binary notation. Perhaps the true inventors of binary arithmetic were English wine merchants!

The first known appearance of binary notation was about 1600 in some unpublished manuscripts of Thomas Harriot (1560-1621). Harriot was a creative man, who came to America with Sir Walter Raleigh; he invented (among other things) the notation now used for "less than" and "greater than" relations; but ill health kept him from publishing many of his discoveries. The first published discussion of the binary system was given in a comparatively little-known work by a Spanish bishop, Juan Caramuel Lobkowitz, who in 1670 discussed the representation of numbers in radices 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, and 60 at some length, but gave no examples of arithmetic operations in nondecimal systems (except for the trivial operation of adding unity).

Ultimately, G. W. Leibnitz wrote an article in 1703 which illustrated binary addition, subtraction, multiplication, and division; this article really brought binary notation into the limelight, and it is usually referred to as the birth of radix 2 arithmetic. Leibnitz later referred to the binary system quite frequently. He did not recommend it for practical calculations, but he stressed its importance in number-theoretical investigations, since patterns in number sequences are often more apparent in binary notation than they are in decimal; he also saw a mystical significance in the fact that everything is expressible in terms of zero and 1.

It is interesting to note that the important concept of negative powers to the right of the radix point was not yet well understood at that time. Leibnitz asked James Bernoulli to calculate π in the binary system, and Bernoulli "solved" the problem by taking a 35-digit approximation to π, multiplying it by 10^{35}, and then expressing this integer in the binary system as his answer. On a smaller scale this would be like saying that $\pi = 3.14$, and $(314)_{10} = (100111010)_2$; hence π in binary is 100111010! The motive for Bernoulli's calculation was apparently to see whether any simple pattern could be observed in this representation of π.

octal

Charles XII of Sweden, whose talent for mathematics perhaps exceeded that of all other kings in the history of the world, hit on the idea of radix 8 arithmetic about 1717. This was probably his own invention, although he had met Leibnitz briefly in 1707. Charles felt radix 8 or 64 would be more convenient for calculation than the decimal system, and he considered introducing octal arithmetic into Sweden; but he died in battle before carrying out such a change.

About 140 years later, a prominent Swedish-American civil engineer named John W. Nystrom decided to carry Charles XII's plans a step further, and he devised a complete system of numeration, weights, and measures based on hexadecimal (radix 16) arithmetic. He wrote, "I am not afraid, or do not hesitate, to advocate a binary system of arithmetic and metrology. I know I have nature on my side; if I do not succeed to impress upon you its utility and great importance to mankind, it will reflect that much less credit upon our generation, upon our scientific men and philosophers." Nystrom devised special means for pronouncing hexadecimal numbers; e.g., $(B0160)_{16}$ was to be read "vybong, bysanton." A similar system, but using radix 8, was proposed about the same time by Alfred B. Taylor. Increased use of the French (metric) system of weights and measures led to extensive debate about the merits of decimal arithmetic during that era.

The binary system was well known as a curiosity ever since Leibnitz's time. It was applied chiefly to the calculation of powers and to the analysis of certain games and puzzles. In 1898, the celebrated Italian mathematician G. Peano showed how to use binary notation as the basis of a "logical" character set of 256 symbols.

Increased interest in mechanical devices for doing arithmetic, especially for multiplication, led several people to consider the binary system for this purpose. A particularly delightful account of this activity is given in the article "Binary Calculation" by E. William Phillips [*Journal of the Institute of Actuaries* 67 (1936), 187-221] together with a record of the discussion which followed a lecture he gave on the subject. Phillips begins by saying, "The ultimate aim of this paper is to persuade the whole civilized world to abandon decimal numeration and to use octonal numeration in its place."

Modern readers of Phillips' article will perhaps be surprised to discover that a radix 8 number system was properly referred to as "octonary" or "octonal," according to all dictionaries of the English language at that time, just as the radix 10 number system is properly called either "denary" or "decimal." The word "octal" did not appear in English language dictionaries until 1961, and it apparently originated as a term for the "base" of a certain class of vacuum tubes. The word "hexadecimal," which has crept into our language even more recently, is a mixture of Greek and Latin stems; more proper terms would be "senidenary" or "sedecimal," or even "sexadecimal," but the latter is perhaps too risqué for computer programmers. One man who attended Mr. Phillips' lecture pointed out a disadvantage of the octal system for business purposes: "5% becomes 3.1463 per 64, which sounds rather horrible."

The first vacuum-tube computer circuits were designed in 1937 by John V. Atanasoff, and the first relay computer circuits were designed independently in the same year by George R. Stibitz. Both men used the binary system for arithmetic in these planned computers, although Stibitz developed excess-3 binary-coded-decimal notation soon afterwards.

The first high-speed computing devices actually built, in the 1940's, used decimal arithmetic. But in 1946, an important memorandum by A. W. Burks, H. H. Goldstine, and J. von Neumann, in connection with the design of the first stored-program computer, gave detailed reasons for their decision to make a radical departure from tradition and to use base-two notation. Since then binary computers have become commonplace. After a dozen years of experience with binary machines, a discussion of the relative advantages and disadvantages of binary notation was given by W. Buchholz in his paper "Fingers or Fists?" [*CACM* 2 (December, 1959), 3-11].

Many interesting variations of positional number systems are possible besides those we have discussed so far. We can, for example, use negative or complex numbers for the radix; or we can use both positive and negative numbers as digits. For the theory and history of these number systems, see *The Art of Computer Programming*, Vol. 2: *Seminumerical Algorithms*, by D. E. Knuth (Addison-Wesley, 1969). References to the original source material from which the historical information in this article has been gathered may also be found in that book. ■

letters

...sed charges are incest and self-perpetuation. Third, the individuals are hardly "casually" selected. It is at least arguable that the skill mix is often wrong, but as one who has been involved in standards recruiting, let me ... you that "casual" is an erroneous charge. Finally, rather than it being the case that the boss doesn't ...ch, it is the usual situation that he ...ches altogether too closely without ...il understanding. In a structure that ... voluntary (at the organization level) management control is quite different ... the "one mistake and out" philosophy espoused, if not actually ef... by business firms.

...e implication that proposed standards are often not published at all is an error. Further, if the readership ...e publications that are willing to publish proposed standards is small or inappropriate, what can be done? ... would DATAMATION publish standards in full, together with an exposition ... discussion of the economic and ...nical consequences of equal or ...ter length?

...e lethargy is, of course, a matter ..., but I dispute your contention that it results from anything other ... your deprecated concept—human ... Typically people try to solve ...ay's problems, curse the past for not ... to make the road easier and deny that they have the time to worry about tomorrow's problems. They forget that tomorrow comes.

Standards solve a second order ...ence and are treated that way. I ... black, so what are the blacks' ... to me? I am not poor, so why ... I pay to fix their needs; I am ... Jew or an Arab, so why should I care whether Egypt (or Israel) has the Bomb; I am not a politician but I could fix things if I were in office; etc., ... etc. The translation is simple. I haven't got time for standards, but if I ... I could do it right.

...ife is complicated, data processing is complicated and standards are complicated. The standardization community is trying to improve its operations. Several groups have been studying the process for the past eighteen months and have, in a coordinated and cooperative manner, generated a set of proposals that cut to the heart of the problem. Whether these proposals will be adopted fully and how much improvement they will generate remains to be seen, but it is certain that there is a recognition of the problems and a serious effort to fix them.

The fact that DATAMATION takes the trouble and space to discuss these questions is encouraging and I hope it continues. At the same time I hope that DATAMATION will not adopt some absolute editorial posture that could result in less than candid commentary on progress. As you know this sort of thing is not unknown in journalism—even technical journalism.

As a final thought, I must protest your implied canard toward *Bradypus tridactylus*. The sloth is a happy, friendly little fellow with soft fur and no pretentions toward imitating a four year old human demolishing a fudgsicle while awaiting Medea's multiple infanticide. Pick on standards all you wish but leave our furry friends alone or I'll have the Sierra Club after you.

T. B. STEEL, JR.
Santa Monica, California

The Editor replies: Some of my best friends are sloths . . . and standards workers.

V15 #4 Datamation

by the numbers

Sir:

Dr. Donald E. Knuth states in his article "The Evolution of Number Systems" in the February issue that the Maya Indians "used a mixed radix system, alternating between radix 20 and radix 18, and so it was not very suitable for operations like multiplication of large numbers, nor is there any known evidence that Mayans were skilled at arithmetic."

The radix 18 was used only in calendar representations in order to be able to carve annual records relatively conveniently onto stone pillars. In all other records and in any calculations, a pure radix 20 notation was used. As for the latter portion of the statement, it would seem to me that a calendar more accurate than our present one, in continuous use without any need of revision for the entire period of Maya civilization, is evidence of a certain amount of skill both in arithmetic and in interpreting their own number system.

PATRICIA NELSON
Syracuse, New York

Dr. Knuth replies: Dr. Nelson has caught me doing some bad scholarship: Four hours of searching in the library today turned up over a dozen sources which confirm her statement about the Maya use of base 18, while I could find none which agreed with what I said. Now I can't remember where I got my information, since I am sure I saw it in two different places.

My remark about arithmetical skill was somewhat overstated; I meant that evidence of multiplication and division has never been found among the Maya. This statement still seems to be substantially correct; but it is hard to make a definite inference about their arithmetical capabilities, since the Spanish missionaries burned all of the "heathen" Maya writings they could find. The Maya did prepare tables of multiples of 91 to aid them in doing certain multiplication problems that arose in connection with

calendar conversion see J. E. S. Thompson, Contr. to Amer. Anthrop., Carnegie Inst., Washington, 1942).

Sir:
Dr. Knuth's article "The Evolution of Number Systems" in the February issue was most interesting. I wish, however, to point out either a misprint or an error in the given value for π. The correct value, to thirty decimal places, is

3.141592653589793
238462643383279

The readers might be interested in the following rhyme, which gives the above digits if one counts the letters in the words. I don't know where the rhyme originated. It was told to me by a friend who had learned it from one of his college professors.

3 1 4 1 5 9 2 6 5
Now I, even I, would celebrate in rhymes inept,
3 5 8 9 7 9 3
The great immortal Syracusan, rivaled nevermore,
2 3 8 4 6 2 6 4
Who in his wondrous lore passed on before,
3 3 8 3 2 7 9
Left men his guidance how to circles mensurate.

OLENA KONIVER
Oxnard, Maryland

Knuth replies: But I know the octonal facts ... 'This gives the octal representation, 3.11037552 ... if "know" is taken to mean no letters!

Sir:
As an amateur, in the literal as well as pejorative sense, of numbering systems, may I express my interest in your writing, and my wonder at needling ask why Dr. Knuth omits entirely from his historical survey the curiosity of Roman numbers?

Surely that system is worth comment, exemplifying as it does both a widely used nonpositional system lacking a zero symbol, and also an operating bi-quinary system.

This lacuna, combined with his lack of mention of the zero as a "step-jump" in numerical manipulations, creates some doubt in one's mind over the remaining six-plus volumes of his projected publication.

P. M. BEATTS
Los Altos, California

negative to photo

Sir:
In your issue of Feb. '69, I read with great interest the MAI 100 Data Transcriber writeup which appeared on Page 149. The story about the transcriber is very good and we are grateful for your consideration in this respect. Unfortunately, however, the picture that appeared in the writeup appears to be one of our competitors, namely Vanguard Data Systems. In addition, I would like to point out that the MAI 100 Data Transcriber is *designed and built* by Digital Information Devices rather than designed by MAI and built by DID as stated in your article.

We at DID are proud of our Data Transcriber and feel that it is considerably more advanced than the competitive equipment for the following reasons: It has a cartridge load using computer compatible ½" tape and reel in the cartridge, automatic threaded loading and unloading, and dual vacuum capstan controls on the tape drive.

We would appreciate very much that, consistent with your accurate and excellent reporting tradition, you would make the necessary correction in a subsequent issue relating to our product.

LEON J. STACIOKAS
President, DID
Norristown, Pennsylvania

Ed. note: See New Products, p. 248.

obfuscation

Sir:
I am putting pen to paper to write you about something that has been annoying me for some time, to wit:

Almost every issue of DATAMATION contains acronyms, abbreviations or just plain nouns of a technical nature which are brand new to me. It is distracting, not to say confusing, to run across such items in the course of reading an article and to have to stop and ponder on their meaning.

I realize that my 17 odd years in business data processing working with only one manufacturer's hardware excludes me from the ranks of the cognoscenti who, without batting an eyelid, recognized the meanings of such items as "unit record equipment," or CRT, or CAI, or a myriad others the first time they saw or heard them.

My main responsibility of writing computer programs for my employer precludes my spending all my time reading technical magazines and attending conferences where so much of this nomenclature first sees the light of day.

So in the interests of making DATAMATION more informative and more pleasant to read, how about either having more footnotes[1] or adding a glossary section devoted to terms that have not before appeared in your magazine.

VLADIMIR V. PRAVIKOFF
Glendale, California

1. As a charter member of CRAMP (Committee for Reasonable Acronyms Mellifluently Phrased), DATAMATION tries to oblige.